Images of Galois representations attached to low weight Siegel modular forms

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Explicit and computational approaches to Galois representations
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The classical case

- $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(N, \epsilon)$ normalised Hecke eigenform, $k \geq 2$

- Associated $\ell$-adic Galois representation

  $$\rho_\ell : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{GL}_2(\overline{\mathbb{Q}_\ell})$$

  unramified for all $p \nmid \ell N$ with

  $$\text{Tr} \rho_\ell(\text{Frob}_p) = a_p, \quad \text{det} \rho_\ell = \epsilon \chi_{\ell}^{k-1}$$

- Associated mod $\ell$ Galois representation

  $$\overline{\rho}_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\overline{\mathbb{F}_\ell})$$
When are $\rho_\ell$ and $\overline{\rho}_\ell$ irreducible?

**Example:** a reducible $\ell$-adic Galois representation

$$G_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty} \sigma_{11}(n)q^n \quad \sim \sim \sim \rightarrow \quad \rho_\ell \cong 1 \oplus \chi_{11}$$

**Theorem (Ribet)**

*If $f$ is cuspidal, then:*

1. $\rho_\ell$ is irreducible for all $\ell$;
2. $\overline{\rho}_\ell$ is irreducible for all but finitely many $\ell$;

**Example:** a reducible mod $\ell$ Galois representation

$$\Delta(z) = 1 + \sum_{n\geq 2} \tau(n)q^n \quad \sim \sim \sim \rightarrow \quad \overline{\rho}_{691} \cong 1 \oplus \overline{\chi}_{691}^{11}$$

**Theorem (Ribet, Momose)**

*If $f$ is not CM, then the image of $\rho_\ell$ is as large as possible for all but finitely many $\ell$.***
“Cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_{\mathbb{Q}}) + \text{ conditions at } \infty$”

- has weights $(k_1, k_2)$, $k_1 \geq k_2 \geq 2$
- has a level $N$
- has a character $\epsilon$
- has Hecke operators $T_p$ and Hecke eigenvalues $a_p$

4 types of cuspidal Siegel modular form:

- General
- Theta lifts/Automorphic inductions
- Saito-Kurokawa/CAP
- Yoshida/endoscopic

\{ reducible Galois representations \}

High weight: $k_2 > 2$
Low weight: $k_2 = 2$
The high weight case: $k_2 > 2$

- Associated $\ell$-adic Galois representation
  \[ \rho_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{\mathbb{Q}_\ell}) \]

  unramified for all $p \nmid \ell N$ with
  \[ \text{Tr} \rho_\ell(\text{Frob}_p) = a_p, \quad \text{sim} \rho_\ell = \epsilon \chi_\ell^{k_1 + k_2 - 3} \]

- Associated mod $\ell$ Galois representation $\overline{\rho}_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{\mathbb{F}_\ell})$

- $\rho_\ell$ is de Rham kinda nice for all $\ell$, and is crystalline nice if $\ell \nmid N$

- The Hecke eigenvalues satisfy the generalised Ramanujan conjecture

**Theorem**

1. *(Ramakrishnan)* If $\rho_\ell$ is nice and $\ell > 2(k_1 + k_2 - 3) + 1$, then $\rho_\ell$ is irreducible;

2. *(BLGGT)* $\overline{\rho}_\ell$ is irreducible for 100% of primes.

3. *(Dieulefait-Zenteno)* The image of $\overline{\rho}_\ell$ contains $\text{Sp}_4(\mathbb{F}_\ell)$ 100% of primes.
The low weight case: $k_2 = 2$

- Associated $\ell$-adic Galois representation

$$\rho_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{\mathbb{Q}_\ell})$$

unramified for all $p \nmid \ell N$ with

$$\text{Tr} \rho_\ell(\text{Frob}_p) = a_p, \quad \text{sim} \rho_\ell = \epsilon \chi_\ell^{k_1-1}$$

- Associated mod $\ell$ Galois representation $\overline{\rho}_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{\mathbb{F}_\ell})$

**Theorem (W.)**

1. If $\rho_\ell$ is nice and $\ell > 2(k_1 - 1) + 1$, then $\rho_\ell$ is irreducible;
2. $\overline{\rho}_\ell$ is irreducible for all but finitely many such primes;
3. The image of $\overline{\rho}_\ell$ contains $\text{Sp}_4(\mathbb{F}_\ell)$ for all but finitely many such primes.
A theoretically checkable condition

Theorem (Jorza)

If \( \ell \nmid N \) and the \( \ell \)-th Hecke polynomial has distinct roots, then \( \rho_\ell \) is nice.

Corollary

If \( \ell > (2k_1 - 1) + 1, \ell \nmid N \), and the \( \ell \)-th Hecke polynomial has distinct roots, then \( \rho_\ell \) is irreducible.

Theorem (W.)

The \( \ell \)-th Hecke polynomial has distinct roots for 100% of primes. Hence, \( \rho_\ell \) is nice for 100% of primes.
Sketch proof for modular forms (Ribet).

If \( f \in S_k(N, \epsilon) \leftrightarrow \rho_\ell \) and \( \rho_\ell \) is reducible then

\[
\text{Kinda nice} \implies \rho_\ell \simeq \psi \oplus \varphi \chi_\ell^{k-1}
\]

1. **CFT**: \( \psi, \varphi \) correspond to Hecke (in this case Dirichlet) characters.
2. Get an equality of partial L-functions

\[
L^*(f \otimes \psi^{-1}, s) = \zeta^*(s)L^*(\varphi \psi^{-1}, s + k - 1);
\]
3. The RHS has a pole at \( s = 1 \), but the LHS is holomorphic.

**Idea**: use the modularity of the subrepresentations of \( \rho_\ell \) to get a contradiction on the automorphic side.
Irreducibility and modularity II

**Idea:** use the modularity of the subrepresentations of $\rho_\ell$ to get a contradiction on the automorphic side.

**Key lemma (W.)**

Either $\rho_\ell$ is irreducible, or it splits as a direct sum of two-dimensional representations which are irreducible, regular and odd.

**Theorem (Taylor)**

*If $\ell$ is sufficiently large, and $\rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}_2(\overline{Q}_\ell)$ is irreducible, regular, crystalline and odd, then $\rho$ is potentially modular.*

- If $\rho_\ell$ is reducible then $\rho_\ell \simeq \sigma_1 \oplus \sigma_2$.
- If $\rho_\ell$ is also crystalline, find automorphic representations $\pi_1, \pi_2$ of $\text{GL}_2(A_K)$ corresponding to $\sigma_1|_K, \sigma_2|_K$.
- Apply a standard $L$-functions argument.
Irreducibility in general

**Conjecture**

If $\pi$ is a cuspidal automorphic representation of $GL_n(A_K)$ then $\rho_\ell$ is irreducible for all primes.

**Known results:**

- $n = 2$: Ribet
- $n = 3$: Blasius-Rogawski if $K$ totally real, $\pi$ essentially self dual

**Partial results:**

- (Barnet-Lamb–Gee–Geraghty–Taylor) if $K$ is CM and $\pi$ is “extremely regular”, then $\rho_\ell$ is irreducible for 100% of primes.
Thank you for listening!